[12-05-11-T10] - Rough Draft

Functions

DEFINITION: FUNCTION. A relation between sets A and B that pairs every element of set A with exactly one element of set B is called a *function* from set A to set B. Set A is called the *domain* of the function, set B is called the *codomain* of the function. The set of elements of B that are paired with those of A is called the *range* of the function. The paired elements are called *ordered* pairs; the element from set A is listed first, the element from set B is listed second. □

The notation that expresses these ideas uses a single letter, often *f*, *g*, or *h*, to name the function. The element from the domain is represented by a single letter; if that letter is x then the element with which x is paired is written $f(x)$. In this example, x is the *argument* of the function f and $f(x)$ is the *value* of f for argument x.

EXAMPLE. The statement "$f : \mathbb{N} \longrightarrow \mathbb{N}$, where $f(x) = x^2$" asserts that the function named f pairs every element of its domain $\mathbb{N}$, the set of positive integers, with an element of its codomain which happens to also be $\mathbb{N}$. The element in $\mathbb{N}$ paired with x is equal to the square of the number x; that is, $f(1) = 1$, $f(2) = 4$, $f(3) = 9$ and so on. The function includes the ordered pairs (1, 1) (2, 4), (3, 9). In this example, the range of f is the set $\{1, 4, 9, 16, 25, \ldots\}$.

THEOREM. f is a function if and only if for every x in its domain, $x_1 = x_2$ implies $f(x_1) = f(x_2)$. Equivealently, f is a function if and only if for every x in its domain, $f(x_1) \neq f(x_2)$ implies $x_1 \neq x_2$. □

EXAMPLE. Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$, where $f(x) = x^2$, is indeed a function.

Solution. Suppose $x_1 = x_2$. Then $f(x_1) = x_1^2 = x_2^2 = f(x_2)$. Since $x_1 = x_2$ implies $f(x_1) = f(x_2)$, f is a function.

EXAMPLE. Show that the equation $x^2 + y^2 = 4$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$, does not define y as a function of x.

Solution. Both $x_1 = 1$, $y_1 = \sqrt{3}$ and $x_2 = 1$, $y_2 = -\sqrt{3}$ satisfy the equation $x^2 + y^2 = 4$, but although $x_1 = x_2$, $y_1 \neq y_2$.

The three figures below illustrate the application of the definition of function.

Figure 1.

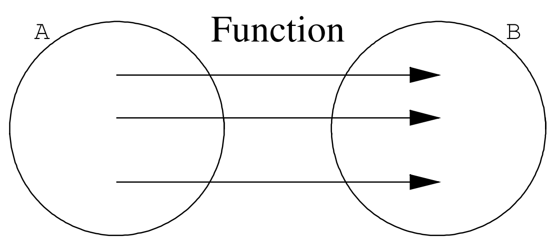

Figure 2.

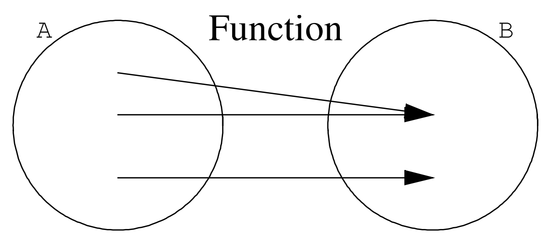

Figure 3.

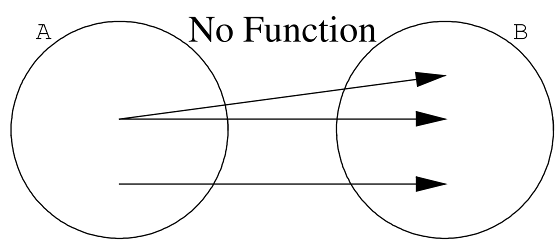

Figure 2 is consistent with the definition of a function. The definition requires that every element of set A be paired with an element of set B. It does not require that each element of A be paired with a different element of B.

Figure 1 depicts an interseting and important case. If every element of A behaves as does the three elements shown, then every element of A is paired with a unique element of B. In this case, the function from A to B is said to be injective or one-to-one or just 1-1. This case is important enough to warrent a definition.

DEFINITION: INJECTIVE (1-1) FUNCTION. A function that pairs every element of a set A with a unique element of a set B is called an *injective* function, or an injection. The terms *1-1* and *one-to-one* are often used instead of the word "injective". □

THEOREM. f is an injective function if and only if for every x in its range, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Equivealently, f is an injective function if and only if for every x in its range, $x_1 \neq x_2$) implies $f(x_1) \neq f(x_2)$. □

THEOREM. A function f has an inverse (is said to be invertable) if and only if f is an injective function. □

THEOREM. A functions f and g are inverse functions if and only if f and g are symmetric with respect to the line $y = x$. □

DEFINITION. A function f is called an *even* function if and only if f is symmetric with respect to the y-axis. Formally, f is an even function if and only if $f(x) = f(-x)$ for all x in the domain of f. □

DEFINITION. A function f is called an *odd* function if and only if f is symmetric with respect to the origin. Formally, f is an odd function if and only if $f(-x) = -f(x)$ for all x in the domain of f. □